Propositional Statements

A mathematical proof is an argument which convinces other people that something is true.

The implication "If p then q" written as $p \Rightarrow q$ means that if p is true, then q must also be true. Statement p is called the premise of the implication and q is called the conclusion.

Direct Proof

In a direct proof, we must assume that p is true, and use definitions, and previous results to deduce that q is also true.

Def: Even and Odd
An even number is an integer of the form $n = 2k$, where k is an integer. (ex. 2, 4, 60, -10, 0)

An odd number is an integer of the form $n = 2k + 1$, where k is an integer.

Any integer must be either odd or even.

Def: Natural Number, Integer, Rational Number, Irrational Number, Real Number
Natural Number $\mathbb{N}$: $0, 1, 2, 3, \ldots$
Integer $\mathbb{Z}$: $\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$
Rational Number $\mathbb{Q}$: A rational number is a number that can be expressed as a fraction p/q where p and q are integers and $q \neq 0$.
Irrational Number $\mathbb{I}$: Numbers that are not rational are called irrational numbers.
Real Numbers $\mathbb{R}$: All rational and irrational numbers are called the real numbers.

Example 1. If x is odd, then $x + 2$ is odd, where $x \in \mathbb{Z}$

Proof:
Know: x is odd so $x = 2n + 1$ for some $n \in \mathbb{Z}$ (n is an integer, whole number)
Want: $x + 2 = 2m + 1$ for some $m \in \mathbb{Z}$

$$x + 2 = (2n + 1) + 2 = 2n + 2 + 1 = 2(n + 1) + 1$$

Now, $n + 1$ is also an integer, call it m, so we have written $x + 2$ as some form of 2 times another integer plus 1. So hence $x + 2$ is also by definition of an odd number, odd.

Example 2. If x is odd and y is even, then $x \cdot y$ is even, where $x, y \in \mathbb{Z}$.

Proof:
Know: x is odd so $x = 2n + 1$ for some $n \in \mathbb{Z}$, and y is even so $y = 2m$ for some $m \in \mathbb{Z}$
Want: $x \cdot y = 2k$ for some $k \in \mathbb{Z}$

$$x \cdot y = (2n + 1)(2m) = (2n)(2m) + 1(2m) = 4nm + 2m = 2(2nm) + 2m = 2(2nm + m)$$

$2nm + m$ is also an integer, call it k, so we have written $x \cdot y$ as some form of 2 times another integer. So hence $x \cdot y$ is also by definition of an even number, even.

Your Turn.

Do the following proofs on the chalkboard in small groups. Reference the examples that we have done together to get the mechanics of the proofs down.

Exercise 3. If x is even, then $x + 40$ is even, where $x \in \mathbb{Z}$.

Exercise 4. If x is odd and y is odd, then $x \cdot y$ is odd, where $x, y \in \mathbb{Z}$.

The things that we are proving now may seem straightforward, so why bother? Right now we are trying to be more comfortable with just knowing what a proof *is*. The exact same proof style is used to prove much more complex, non-straightforward, non-trivial results in math!

Example 5. If x is even, then x^4 is even, where $x \in \mathbb{Z}$.

Proof:
Know: x is even so $x = 2n$ for some $n \in \mathbb{Z}$
Want: $x^4 = 2k$ for some $k \in \mathbb{Z}$

$$x^4 = (2n)^4 = (2n)(2n)(2n)(2n) = 16n^4 = 2(8n^4)$$

we know $8n^4 \in \mathbb{Z}$, so set $8n^4 = k$, and done! We have proved that x^4 is an even number!

Example 6. If $q, r \in \mathbb{Q}$ (are rational), then $q + r \in \mathbb{Q}$ (is rational).

Proof:
Know: q, r are even so $q = \frac{a}{b}$ and $r = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$, where $b, d \neq 0$
Want: $q + r = \frac{f}{g}$ for some $f, g \in \mathbb{Z}$, where $g \neq 0$

$$q + r = \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

Now, $ad + bc, bd \in \mathbb{Z}, bd \neq 0$. So we have written $q + r$ in the form of an integer divided by a nonzero integer, so by definition $q + r$ is a rational number.

Your Turn.

Exercise 7. If x is odd, then $x^3 + 5x + 1$ is odd, where $x \in \mathbb{Z}$.

Exercise 8. If $q, r \in \mathbb{Q}$, then $q \cdot r \in \mathbb{Q}$.

Exercise 9. If $q \in \mathbb{Q}$, then $-q \in \mathbb{Q}$.

Proof by Contraposition

Example 1. If $7x + 9$ is even, then x is odd, where $x \in \mathbb{Z}$

We know how to do direct proofs. This problem smells like what we have just done with direct proofs. Lets attempt that here.

Attempted Proof:

Know: $7x + 9$ is even so $7x + 9 = 2n$ for some $n \in \mathbb{Z}$
Want: x is odd so $x = 2m + 1$ for some $m \in \mathbb{Z}$

$$7x + 9 = 2n \;\rightarrow\; 7x = 2n - 9 \;\rightarrow\; x = \frac{2n - 9}{7}$$

Trouble: that horrible mess $\frac{2n-9}{7}$ doesn't smell like something of the form $2m + 1$...

Sometimes, it is better to (and sometimes we need to) assume that the opposite of the result is true, and conclude the opposite of our assumption is true.

In a propositional statement, we have that the contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. These two statements are **equivalent**. This means that if you can prove one of the statements, the other one must also be true.

Proof by Contraposition:
Suppose: x is not odd so it is even, meaning $x = 2n$ for some $n \in \mathbb{Z}$
Want: $7x + 9$ is not even so it is odd, meaning $7x + 9 = 2m + 1$ for some $m \in \mathbb{Z}$

$$x = 2n \;\rightarrow\; 7x = 7(2n) = 14n \;\rightarrow\; 7x + 9 = 14n + 9 = 2(7n + 4) + 1$$

Done! Why?

Def: $<, >, \leq, \geq$
Students: What does an inequality mean in terms of a number line? Help me draw it!

Example 2. If $n^2 > 100$, then $n > 10$, where $n \in \mathbb{N}$ (n is a natural number, which is a positive integer or 0)

Proof by Contraposition:
Suppose: $n \not> 10$, so n is at most 9, Why? See number line.
Want: $n^2 \not> 100$, so n is at most 99, $n^2 \leq 99$

$$n \leq 9 \;\rightarrow\; n^2 \leq 81 < 99 \;\rightarrow\; n^2 \leq 99$$

Note that n^2 can only be at most 81, but we have rounded up here to being at most 99. This is a legal move.

Your Turn.

Exercise 3. If $3x + 2$ is odd, then x is odd, where $x \in \mathbb{Z}$.

Exercise 4. If $x^2 - 6x + 5$ is even, then x must be odd, where $x \in \mathbb{Z}$.

Exercise 5. If r is irrational, then $-r$ is also irrational ($-r \in \mathbb{I}$). (recall that an irrational number is defined to be a number that is NOT rational).

Exercise 6. If $n^3 > 64$, then $n > 4$, where $n \in \mathbb{N}$ (n is a natural number, which is a positive integer or 0)

Proof by Contradiction

Given $p \rightarrow q$, suppose that q is not true and p is true to deduce that this is impossible. In other words, we want to show that it is impossible for our hypothesis to occur but the result to not occur. We always begin a proof by contradiction by supposing that q is not true ($\neg q$) and p is true.

Example 1. If x^2 is odd, then x is odd, where $x \in \mathbb{Z}$

Proof by Contradiction:
Suppose: x is not odd so x is even where $x = 2n$, and x^2 is odd.
Want: a contradiction somewhere.

$$x = 2n \;\rightarrow\; x^2 = (2n)^2 = 4n^2 = 2(2n^2) \;\; X$$

But $2n^2$ is an integer, so we have written x^2 as a form of 2 times an integer, and thus by definition x^2 must be even. This contradicts our assumption that x^2 was odd.

Your Turn.

Exercise 2. If x^3 is even, then x is even, where $x \in \mathbb{Z}$.

Exercise 3. If p is irrational and q is rational, then $p + q$ is irrational.

Example 4. If $7x + 9$ is even, then x is odd, where $x \in \mathbb{Z}$

We can prove this using either a proof by contradiction or a proof by contradiction. This is not special coincidence! In general, any proof by contraposition can be written as a proof by contradiction.

Proof by Contradiction:
Help me!

Proof by Contraposition:
Help me!

So in general, why can any proof by contraposition can be written as a proof by contradiction?

Example 5 (if time): Prove that $\sqrt{2}$ is irrational.

Proof: Suppose $\sqrt{2}$ is rational, i.e. $\sqrt{2} = \frac{a}{b}$ for some integers a and b with $b \neq 0$. We will reduce the fraction $\frac{a}{b}$ to its simplest form. Squaring both sides of the equation $\sqrt{2} = \frac{a}{b}$ and multiplying both sides by b^2, we get $a^2 = 2b^2$. Thus a^2 is even by definition, and a is even (why?). Thus $a = 2k$ for some integer k, so $a^2 = 4k^2$, and hence $b^2 = 2k^2$. Thus b^2 is also even by definition, so b is even. Since a and b are both even, $a/2$ and $b/2$ are integers. and $\sqrt{2} = \frac{a/2}{b/2}$, because $\frac{a/2}{b/2} = \frac{a}{b}$. But we said before $\frac{a}{b}$ is in its simplest form and cannot be reduced. We just reduced $\frac{a}{b}$ by a factor of 2, so this is a contradiction. X
Therefore $\sqrt{2}$ cannot be rational.

Conclusion

All 3 basic proof strategies can be used for the same problem. However, it may be better (or only possible) to use one proof type over the others for certain problems!

If and Only If Statements

Previously, we have just proved that "If x^2 is odd, then x is odd". Is it also true that "If x is odd, then x^2 is odd"? YES! But convince me!

When this happens where $p \rightarrow q$ and $q \rightarrow p$, we have an **if and only if** statement.

Example 1. x^2 is odd if and only if x is odd, where $x \in \mathbb{Z}$ (x^2 odd $\iff$ x is odd)

In words:
($\Rightarrow$)x^2 is odd if x is odd, and
($\Leftarrow$)x^2 is odd ONLY if x is odd

This is a bidirectional statement (notice the 2 way arrow!) To prove this, we simply need to prove the two implications ($\Rightarrow$) and ($\Leftarrow$), namely we need to prove that

($\Rightarrow$) If x^2 is odd, then x is odd.
This part is Done! We already proved this with a proof by contradiction.

($\Leftarrow$) If x is odd, then x^2 is odd.
Help me! What type of proof should we use?

Note: For an if and only if statement, always write out both directions/implications that you want to prove. An if and only if proof is nothing more than just 2 proofs in one.

Your Turn.

Exercise 2. x is odd $\iff$ x^3 is odd, where $x \in \mathbb{Z}$.

Exercise 3. x is even $\iff$ $7x + 4$ is even, where $x \in \mathbb{Z}$.

Proof by Cases

A "Proof by cases" is best explained by an example. It is basically multiple proofs in one that uses direct proofs, proofs by contrapostion, and/or proofs by contradiction.

Example 1. For every integer x, the integer $x(x + 1)$ is even.

Proof: Given x, we have 2 options in life because x must be either odd or even.

Case I: x is odd, so $x = 2n + 1$ for some $n \in \mathbb{Z}$. We want $x(x + 1) = 2m$ for some $m \in \mathbb{Z}$

$$x(x + 1) = (2n + 1)(2n + 1 + 1) = (2n + 1)(2n + 2) = 4n^2 + 6n + 2 = 2(2n^2 + 3n + 1)$$

Done! Why?

Case II: x is even, so $x = 2n$ for some $n \in \mathbb{Z}$. We want $x(x + 1) = 2m$ for some $m \in \mathbb{Z}$

$$x(x + 1) = (2n)(2n + 1) = 4n^2 + 2n = 2(2n^2 + n)$$

Done! Why?

Therefore, we have checked that given ANY x integer value, $x(x + 1)$ is even. Note that this worked because we know that any integer x MUST either be odd or even. So we have 2 options in life, and we checked both options to see that the conclusion $x(x + 1) = 2m$ will hold.

Example 2. For every integer x, $x + 204$ has the same **parity** as x.

Def: Parity
a and b are said to have the same parity if both are even or both are odd.

Proof: Given x, we have 2 options in life and x must be either odd or even.

Case I: x is odd, so $x = 2n + 1$ for some $n \in \mathbb{Z}$. We want $x + 204 = 2m + 1$ for some $m \in \mathbb{Z}$

$$x + 204 = 2n + 1 + 204 = 2n + 205 = 2(n + 102) + 1$$

Done! Why?

Case II: Help me! x is even, so $x = 2n$ for some $n \in \mathbb{Z}$. We want $x + 204 = 2m$ for some $m \in \mathbb{Z}$

$$x + 204 = 2n + 204 = 2(n + 102)$$

Done! Why?

Your Turn.

Exercise 3. For every integer x, x^2 has the same parity as x