Propositional Logic Solutions

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1. Given n propositional statements, how many possible combinations of their states exist? In the case we have exactly two propositions P and Q ($n = 2$), what are the possible combinations?

   Solution: 2^n**;** $P \wedge Q, P \wedge \neg Q, \neg P \wedge Q, \neg P \wedge \neg Q$**.**

2. Convert the following sentences to expressions in propositional logic. You should need at most three propositions per statement. Please specify what the propositions are.

   (a) It rains.
   Solution: $P :=$ **It rains;** P**.**

   (b) It rains or it doesn't rain.
   Solution: $P :=$ **It rains;** $P \vee \neg P$**.**

   (c) If martians exist, they have red hair.
   Solution: $P :=$ **Martians exist,** $Q :=$ **Martians have red hair;** $P \implies Q$**.**

   (d) He has an Ace only if he does not have a Knight or a Spade
   Solution: $P :=$ **He has a knight,** $Q :=$ **He has a Spade,** $R :=$ **He has an Ace;** $\neg(P \vee Q) \implies R$**.**

   (e) New allocation not being worse than initial endowment is a necessary and sufficient condition for Pareto efficiency.
   Solution: $P :=$ **New allocation is worse that initial endowment,** $Q :=$ **Pareto efficiency exists;** $\neg P \Leftrightarrow Q$**.**

   (f) If independence implies uncorrelatedness, correlatedness implies dependence.
   Solution: $P :=$ **Independence,** $Q :=$ **Uncorrelatedness;** $(P \Rightarrow Q) \Rightarrow$

$(\neg Q \Rightarrow \neg P)$**. Alternatively,** $P :=$ **Dependence,** $Q :=$ **Correlatedness;** $(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P)$**.**

3. Which of the following are valid deductions?

   (a) Assertion: If I leave my umbrella at home, I will get wet today.
   Hypothesis 1: I got wet, so I must have left my umbrella at home.
   Hypothesis 2: I didn't leave my umbrella at home, so I will not get wet today.
   Hypothesis 3: I did not get wet today, so I must not have left my umbrella at home.
   Solution: In propositional form, $P :=$ **I leave my umbrella at home,** $Q :=$ **I get wet. Assertion:** $P \Rightarrow Q$**, Hypothesis 1:** $Q \Rightarrow P$**, Hypothesis 2:** $\neg P \Rightarrow \neg Q$**, Hypothesis 3:** $\neg Q \Rightarrow \neg P$**. Hypothesis 3 is the only logical deduction.**

   (b) Assertion: If two random variables X and Y are independent, their correlation is zero.
   Hypothesis 1: X and Y have zero correlation, so they are independent.
   Hypothesis 2: X and Y don't have zero correlation, so they must not be dependent.
   Hypothesis 3: X and Y are dependent, so their correlation is not zero.
   Solution: This is nearly identical to part (a). $P := X$ **and** Y **are independent,** $Q := X$ **and** Y **have zero correlation. Assertion:** $P \Rightarrow Q$**, Hypothesis 1:** $Q \Rightarrow P$**, Hypothesis 2:** $\neg Q \Rightarrow \neg P$**, Hypothesis 3:** $\neg P \Rightarrow \neg Q$**. Hypothesis 2 is the only logical deduction.**

4. Simplify the following Boolean expression such that it only has three boolean operators:

$$\neg(A \vee \neg C) \vee ((\neg A \wedge B) \vee B)$$

Solution:

$$\begin{aligned}
& \neg(A \vee \neg C) \vee ((\neg A \wedge B) \vee B) \\
\textbf{Using De Morgan's Law: } & \Leftrightarrow (A \wedge \neg\neg C) \vee ((\neg A \wedge B) \vee B) \\
\textbf{Double negations cancel: } & \Leftrightarrow (A \wedge C) \vee ((\neg A \wedge B) \vee B) \\
\textbf{Identity law: } & \Leftrightarrow (A \wedge C) \vee ((\neg A \wedge B) \vee (B \wedge 1)) \\
\textbf{Distributive law: } & \Leftrightarrow (A \wedge C) \vee (B \wedge (\neg A \vee 1)) \\
\textbf{Annulment Law: } & \Leftrightarrow (A \wedge C) \vee (B \wedge 1) \\
\textbf{Identity law: } & \Leftrightarrow (A \wedge C) \vee B
\end{aligned}$$

Constructing a truth table verifies our solution.

5. Show that $P \implies Q$ is an equivalent statement to $\neg P \vee Q$ by filling in the following truth table:

Solution:

P	Q	$\neg P$	$\neg P \vee Q$	$P \implies Q$
1	**1**	**0**	**1**	**1**
1	**0**	**0**	**0**	**0**
0	**1**	**1**	**1**	**1**
0	**0**	**1**	**1**	**1**

6. Fill in the following truth table. Which binary operation have you derived in the last column? (From $\vee$, $\wedge$, $\implies$, $\iff$)

Solution:

P	Q	$\neg P$	$\neg P \vee Q$	$\neg Q$	$\neg Q \vee P$	$(\neg P \vee Q) \wedge (\neg Q \vee P)$
1	**1**	**0**	**1**	**0**	**1**	**1**
1	**0**	**0**	**0**	**1**	**1**	**0**
0	**1**	**1**	**1**	**0**	**0**	**0**
0	**0**	**1**	**1**	**1**	**1**	**1**

7. What is the boolean expression equivalent to "MUX Out" below? Draw the circuit that corresponds with this expression. The circuit is known as a multiplexer.

P	Q	S	MUX Out
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

Solution: If we can figure out when the output is non-zero, we know the expression. So we consider all the rows with a 1 **as their out, and "sum"**

them:

$$(P \land Q \land S) \lor (P \land Q \neg S) \lor (P \land \neg Q \land S) \lor (\neg P \land Q \land \neg S)$$

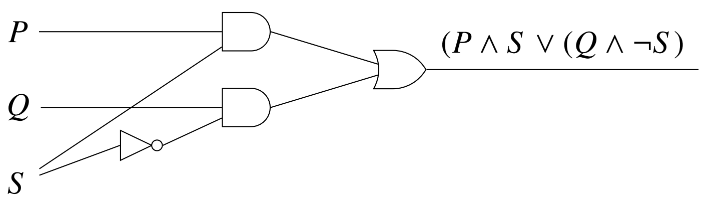
Rearranging via commutativity of disjunction:

$$\Leftrightarrow (P \land Q \land S) \lor (P \land \neg Q \land S) \lor (P \land Q \land \neg S) \lor (\neg P \land Q \land \neg S)$$

Distributivity: $\Leftrightarrow (((P \land Q) \lor (P \land \neg Q)) \land S) \lor (((P \land Q)) \lor (\neg P \land Q) \neg S)$

Distributivity: $\Leftrightarrow (P \land (Q \lor \neg Q) \land S) \lor (Q \land (P \lor \neg P) \land \neg S)$

Complement: $\Leftrightarrow (P \land S) \lor (Q \land \neg S)$



8. Prove that $(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow \bot) \Rightarrow (P \Rightarrow \bot))$. The consequent is known as the contrapositive.

Solution: Note that in natural deduction style proofs, statements above horizontal lines are assumptions. So lines $1, 2$ and 3 are assumptions, while lines 4 and 5 are deductions from the assumptions (Modus Ponens):

1	$P \Rightarrow Q$
2	$\quad Q \Rightarrow \bot$
3	$\quad\quad P$
4	$\quad\quad Q$
5	$\quad\quad \bot$
6	$\quad P \Rightarrow \bot$
7	$(Q \Rightarrow \bot) \Rightarrow (P \Rightarrow \bot)$
8	$(P \Rightarrow Q) \Rightarrow ((Q \Rightarrow \bot) \Rightarrow (P \Rightarrow \bot))$